LETTERS

Interchange modes in a collisional plasma

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Gross plasma stability can derive from plasma compressibility in the bad curvature regions of closed field line systems. In this situation magnetohydrodynamic (MHD) theory predicts that the maximum pressure gradient that is stable is proportional to γ, the ratio of specific heats. This article will examine the accuracy of the MHD prediction for electrostatic interchange modes using kinetic theory. The maximum sustainable pressure gradient is found to be dependent on the ratio of the temperature and density gradients ($\eta \equiv (n/T)(\nabla T/\nabla n)$) as well as on the ion gyro-radius scale length. For $\eta = 2/3$ the MHD stability condition is reproduced. When η deviates from 2/3 the mode changes character and the stability criterion becomes more stringent. © *2000 American Institute of Physics.* [S1070-664X(00)04309-3]

Closed field line systems, such as a levitated dipole, provide a promising new approach for the magnetic confinement of plasmas for controlled fusion.[1,2] The plasma in a closed field line system can be stabilized in so-called ''bad curvature'' regions by plasma compressibility.

In magnetohydrodynamic (MHD) theory stability by compressibility limits the pressure gradient to a value that is proportional to the ratio of specific heats, γ ($\gamma = 5/3$ in three-dimensional systems). This comes about as a result of an assumed simple form for the plasma equation of state. In addition, MHD theory only concerns the pressure gradient and does not indicate the destabilizing effects that can derive from the independent variation of the density and temperature gradient profiles.

To refine the predicted stability limit we use a kinetic theory approach and consider the marginal stability for low beta (electrostatic) flute/interchange modes. We will derive the stability boundary for electrostatic interchange modes in a collisional plasma that includes the effect of the relative temperature and density gradient and also includes finite Larmor radius (FLR) corrections. This treatment uses the appropriate collisionality regime for the LDX experiment[2] and provides a valuable comparison between the stability properties predicted by ideal MHD and the richer (and more accurate) prediction of kinetic theory.

In MHD theory the stability of interchange modes at arbitrary beta requires[3,4]

$$\frac{1}{p}\frac{dp}{d\psi} \leq \frac{2\gamma\langle \kappa_\psi \rangle}{1+\gamma\langle \beta \rangle/2}, \tag{1}$$

with the flux tube average defined as

$$\langle a \rangle \equiv \frac{\oint a\,dl/B}{\oint dl/B}. \tag{2}$$

The curvature $\boldsymbol{\kappa} = \kappa_\psi \nabla \psi$ and $\gamma = 5/3$. We will compare the MHD prediction with the predictions of the more general plasma drift kinetic equation. We will utilize the electrostatic limit of the drift kinetic equation. To compare the MHD result with kinetic theory we define

$$\hat{\omega}_{*p} \equiv \frac{k_\perp \nabla p}{\Omega_i m_i n_i} \tag{3}$$

and

$$\hat{\omega}_d^{\text{mhd}} \equiv \frac{2}{e}(k_\perp R)T_i \frac{\oint dl\,\kappa/RB^2}{(1+\gamma\langle \beta \rangle/2)\oint dl/B} \tag{4}$$

with R the cylindrical radial coordinate and $k_\perp R = m \gg 1$. The MHD stability requirement [Eq. (1)] can therefore be written as

$$\hat{\omega}_{*p} \leq \gamma\hat{\omega}_d^{\text{mhd}}. \tag{5}$$

Following the treatment of Lane[5] we will consider the solution of the drift kinetic equation in the high collision frequency limit (for both ion and electron species). We therefore apply the ordering for both ions and electrons:

$$\Omega_c > \omega_b > \nu > \omega_* \sim \omega_d \sim \omega, \tag{6}$$

with Ω_c the cyclotron frequency, ω_b the bounce frequency, ν the collision frequency, ω_* the diamagnetic drift frequency and ω_d the magnetic curvature drift frequency.

To derive the stability criterion for electrostatic modes we consider a fluctuating potential (ϕ) and ignore any equilibrium electrostatic potential. From Faraday's law it is possible for a perturbation to leave the magnetic field undisturbed if $E = -\nabla\phi$ (which is consistent with $\beta \ll 1$).

The drift kinetic equation was derived under the assumption that the wave frequency ω is less than the cyclotron frequency Ω_c and that the ion Larmor radius ρ_i is shorter than the perpendicular wavelength $\lambda = 2\pi/k_\perp$ which is, in turn, short compared to a parallel wavelength, $2\pi/k_\parallel$. The appropriate equation for the distribution function $\tilde{f}$ is then[6–8]

$$\tilde{f} = q\phi F_{0\epsilon} + J_0(k_\perp\rho)h, \tag{7}$$

and the nonadiabatic response h satisfies

$$(\omega - \omega_d + iv_\parallel \mathbf{b}\cdot\nabla')h = -(\omega - \omega_*)q\phi F_{0\epsilon}J_0(k_\perp\rho)$$
$$+ iC(h). \tag{8}$$

In Eq. (8) $C(h)$ is the collision operator, $J_0(k_\perp\rho)$ is the Bessel function of the first kind, $F_0(\epsilon, \psi)$ is the equilibrium distribution function, i.e.,

$$F_0 = \left(\frac{m}{2\pi T}\right)^{3/2} n_0 e^{-\epsilon/T} \tag{9}$$

and

$$F_{0\epsilon} \equiv \frac{\partial F_0}{\partial\epsilon},$$

$$\omega_* = \frac{\mathbf{b}\times\mathbf{k}_\perp\cdot\nabla' F_0}{m\Omega_c F_{0\epsilon}},$$

$$\omega_d = \mathbf{k}_\perp\cdot\mathbf{b}\times\frac{(v_\parallel^2\mathbf{b}\cdot\nabla\mathbf{b} + \mu\nabla B)}{\Omega_c}$$

$$= \mathbf{k}_\perp\cdot\mathbf{b}\times\frac{\epsilon(2(1 - \lambda B)\mathbf{b}\cdot\nabla\mathbf{b} + \lambda\nabla B)}{\Omega_c}, \tag{10}$$

$$\mathbf{B} = \nabla\psi\times\nabla\theta, \quad \mathbf{b} = \mathbf{B}/|\mathbf{B}|.$$

We have defined $\epsilon = v^2/2$, $\mu = v_\perp^2/2B$ and $\lambda = \epsilon/\mu$. The gradient ∇' is taken at constant ϵ and μ. The magnetic flux function is ψ and θ is the azimuthal angle. Assuming $k_\perp\rho \ll 1$ yields

$$J_{0j} \approx 1 - \frac{(k_\perp\rho_j)^2}{2}\frac{\lambda B_0^2}{B}\frac{\epsilon}{T} \tag{11}$$

with the gyro-radius ρ_j, defined as $\rho_j^2 = T/m\Omega_0^2$, B_0 the magnetic field at the field minimum and Ω_0 the associated cyclotron frequency at the location $R = R_0$.

We consider perturbations whose growth time is long compared to a particle bounce time in Eq. (8) and obtain

$$v_\parallel \mathbf{b}\cdot\nabla h \approx 0, \tag{12}$$

i.e., $h = h_0(\epsilon, \mu, \psi)$ a constant along a field line. We will determine the constant h_0 by taking the bounce average of Eq. (3),

$$(\omega - \bar{\omega}_d)h_0 = -(\omega - \omega_*)q\overline{\phi J_0}F_{0\epsilon} + i\bar{C}(h_0). \tag{13}$$

The overbar indicates a bounce time average:

$$\bar{\phi} = \frac{1}{\tau_b}\oint\frac{\phi(l)dl}{\sqrt{1 - \lambda B}}, \tag{14}$$

with the bounce time, τ_b, defined as

$$\tau_b = \oint\frac{dl}{\sqrt{1 - \lambda B}}. \tag{15}$$

We will assume that the collision operator conserves particles and energy. With the chosen ordering and these assumed conservation properties the exact form of the collision operator does not enter the results.

We will analyze Eq. (13) in the high collisionality limit ($\nu/\omega \gg 1$) and write $h = h_0 + h_1 + \cdots$. To lowest order we find that h_0 is proportional to a Maxwellian distribution function, F_0, multiplied by a perturbed density, δn, and having a perturbed temperature $T + \delta T$.[5] Expanding to first order we obtain

$$h_0 = \delta n\left(\frac{m}{2\pi(T + \delta T)}\right)^{3/2} e^{-\epsilon/(T + \delta T)}$$

$$\approx \left[\frac{\delta n}{n_0} + \frac{\delta T}{T}\left(\frac{\epsilon}{T} - \frac{3}{2}\right)\right]F_0. \tag{16}$$

To next order the drift kinetic equation becomes:

$$(\omega - \bar{\omega}_d)(h_0 + h_1) = -(\omega - \omega_*)q\overline{\phi J_0}F_{0\epsilon} + i\bar{C}(h_1). \tag{17}$$

We can obtain the perturbed density for electron (ion) species by taking the velocity space average, defined by:

$$\int g(\mathbf{v})d^3v = \pi\left(\frac{2}{m}\right)^{2/3}B\int_0^\infty d\epsilon\,\epsilon^{3/2}\int_0^{1/B}d\lambda\frac{g}{\sqrt{1 - \lambda B}}. \tag{18}$$

The collision operator conserves particles and energy and therefore the flux tube and velocity average will annihilate it, i.e.,

$$\int dl/B\int d^3v\,\bar{C}(h) = \int dl/B\int d^3v(\epsilon/T - 3/2)\bar{C}(h) = 0. \tag{19}$$

We will define $\hat{\omega}_*$ by

$$\hat{\omega}_{*j} = \frac{T\mathbf{k}_\perp\times\mathbf{b}\cdot\nabla n_0}{n_j m\Omega}, \tag{20}$$

and write

$$\omega_* = \hat{\omega}_*(1 + \eta(\epsilon/T - 3/2))F_0, \tag{21}$$

with $\eta = d\ln T/d\ln n$. Notice that $\hat{\omega}_{*p} = \hat{\omega}_*(1 + \eta)$. Taking the flux tube and velocity space average of Eq. (17) and using Eq. (2) we obtain the following expression for the non-adiabatic perturbed density δn_j for species j:

$$\delta n_j = \{\hat{\omega}_{dj}n_0(\delta T/T) + (q_j/T_j)[(\omega - \hat{\omega}_*)(\bar{\phi} - K_j/2)$$

$$+ \hat{\omega}_{*j}\eta_j K_j/2]\}/(\omega - \hat{\omega}_{dj}), \tag{22}$$

with $\hat{\omega}_d$ defined as

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$$\hat{\omega}_d = \frac{cT(Rk_\perp)}{qV} \int \frac{dl}{B^2 R}(\kappa + \nabla B/B). \tag{23}$$

In the low β, $\kappa \approx \nabla B/B$ and $\hat{\omega}_d$ becomes equal to the MHD definition given in Eq. (4).

The finite-Larmor-radius (FLR) parameter is K_j,

$$K_j = (B_0 R_0)^2 (k_{\perp 0} \rho_{j0})^2 \frac{\int \phi \, dl/(B^3 R^2)}{\int dl/B}, \tag{24}$$

with $k_{\perp 0} \rho_{j0}$ evaluated on the outer midplane.

To obtain $(\delta T_j/T_j)$ we take the flux tube average and integrate over velocity space for $(\epsilon/T - 3/2) \times$ Eq. (17) (which again annihilates the collision operator) to obtain

$$\delta T_j/T_j = \{\tfrac{2}{3}(\delta n_j/n_j)\hat{\omega}_{dj} - (q_j/3T)[(\omega - \hat{\omega}_{*j} - \tfrac{7}{2}\hat{\omega}_{*j}\eta_j)K_j$$
$$+ 3\bar{\phi}\eta_j\hat{\omega}_{*j}]\}/(\omega - \tfrac{7}{3}\hat{\omega}_{dj}). \tag{25}$$

Equation (22) together with Eq. (25) yields the no-adiabatic perturbed density response of species j:

$$\frac{\delta n_j}{n} = \frac{q_j \bar{\phi}}{T_j} \left[\frac{\omega^2 - \omega(\tfrac{7}{3}\hat{\omega}_{dj} + \hat{\omega}_{*j}) + \hat{\omega}_{dj}\hat{\omega}_{*j}(\tfrac{7}{3} - \eta_j)}{\omega^2 - \tfrac{10}{3}\omega\hat{\omega}_{dj} + \tfrac{5}{3}\hat{\omega}_{dj}^2} \right]$$
$$+ \frac{q_j K_j}{T_j} \left[\frac{-\tfrac{1}{2}\omega^2 + \omega(\tfrac{5}{6}\hat{\omega}_{dj} + \tfrac{1}{2}\hat{\omega}_{*j}(1+\eta_j)) - \tfrac{5}{6}\hat{\omega}_{dj}\hat{\omega}_{*j}}{\omega^2 - \tfrac{10}{3}\omega\hat{\omega}_{dj} + \tfrac{5}{3}\hat{\omega}_{dj}^2} \right]$$
$$\equiv \frac{q_j}{T_j}[\bar{\phi}G_{1j}(\omega, \hat{\omega}_{*j}, \hat{\omega}_{dj}) + K_j G_{2j}(\omega, \hat{\omega}_{*j}, \hat{\omega}_{dj})], \tag{26}$$

and the temperature perturbation $\delta T/T$:

$$\frac{\delta T_j}{T_j} = \frac{q_j \bar{\phi}}{T_j} \left[\frac{\omega(\tfrac{2}{3}\hat{\omega}_{dj} - \eta_j\hat{\omega}_{*j}) - \hat{\omega}_{dj}\hat{\omega}_{*j}(\tfrac{2}{3} - \eta)}{\omega^2 - \tfrac{10}{3}\omega\hat{\omega}_{dj} + \tfrac{5}{3}\hat{\omega}_{dj}^2} \right]$$
$$+ \frac{q_j K_j}{T_j} \left[\frac{-\tfrac{1}{3}\omega^2 + \omega(\tfrac{1}{3}\hat{\omega}_{*j} + \tfrac{7}{6}\eta_j\hat{\omega}_{*j}) - \tfrac{5}{6}\eta_j\hat{\omega}_{dj}\hat{\omega}_{*j}}{\omega^2 - \tfrac{10}{3}\omega\hat{\omega}_{dj} + \tfrac{5}{3}\hat{\omega}_{dj}^2} \right]$$
$$\equiv \frac{q_j}{T_j}[\bar{\phi}H_{1j}(\omega, \hat{\omega}_{*j}, \hat{\omega}_{dj}) + K_j H_{2j}(\omega, \hat{\omega}_{*j}, \hat{\omega}_{dj})]. \tag{27}$$

We apply quasi-neutrality to obtain a dispersion relationship and we multiply the quasi-neutrality equation by ϕ^* and take a flux tube average to obtain

$$0 = \sum_j q_j \int \frac{dl}{B} \int d^3 v \left(\frac{q_j \phi^2}{T} F_0 + \phi^* J_0(k_\perp \rho_j) h_{0j} \right). \tag{28}$$

We assume that the modes are flutelike, i.e., $\overline{\phi^2} = \bar{\phi}^2$, and that the electron finite Larmor radius corrections are insignificant, i.e., $K_e \sim 0$. This yields a dispersion relation of the following form:

$$0 = -2 + G_{1e} + G_{1i} + \kappa_i \left(G_{2i} - \frac{G_{1i}}{2} - \frac{H_{1i}}{2} \right). \tag{29}$$

The finite-Larmor-radius parameter κ_i becomes (for flute modes)

$$\kappa_i = (B_0 R_0)^2 (k_{\perp 0} \rho_{i0})^2 \frac{\int dl/(B^3 R^2)}{\int dl/B}. \tag{30}$$

Equation (29) is a quartic equation with four roots. Assuming $\hat{\omega}_{*i} = -\hat{\omega}_{*e}$ and $\hat{\omega}_{di} = -\hat{\omega}_{de}$ we obtain

$$a_4\omega^4 + a_3\omega^3 + a_2\omega^2 + a_1\omega + a_0 = 0, \quad a_4 = 9\kappa_i,$$

$$a_3 = -15\kappa_i\hat{\omega}_{de} + 9(1+\eta)\kappa_i\hat{\omega}_{*e},$$

$$a_2 = -5(6+7\kappa_i)\hat{\omega}_{de}^2$$
$$\qquad + 3(6+6\eta - 5\kappa_i - 10\eta\kappa_i)\hat{\omega}_{de}\hat{\omega}_{*e}, \tag{31}$$

$$a_1 = 25\kappa_i\hat{\omega}_{de}^3 - 5(7-3\eta)\kappa_i\hat{\omega}_{de}^2\hat{\omega}_{*e},$$

$$a_0 = 50\omega_{de}^4 + 5(-14+6\eta+5\kappa_i)\hat{\omega}_{de}^3\hat{\omega}_{*e}.$$

In the limit that $\omega \gg \hat{\omega}_* \sim \hat{\omega}_d$, Eq. (29) yields the MHD dispersion relation, namely,

$$\omega^2 = -\frac{\hat{\omega}_*\hat{\omega}_d(1+\eta)}{\kappa_i} + \frac{10}{3}\frac{\hat{\omega}_d^2}{\kappa_i}, \tag{32}$$

with $\hat{\omega}_* \equiv 2\hat{\omega}_{*e}$. There can be a fast growing instability when $\hat{\omega}_d < 0$ and $\hat{\omega}_* < 0$. The first term in Eq. (32) is the MHD growth rate and the second term is the stabilizing compressibility. Equation (32) assumes $\omega \gg \hat{\omega}_*$ and therefore cannot be used to predict marginal stability.

When $\kappa_i = 0$, $a_4 = a_3 = 0$, and we obtain a temperature gradient driven drift wave. The dispersion relation for this mode becomes

$$\omega^2 = \hat{\omega}_{de}^2 \frac{5\omega_{de} - (7-3\eta)\omega_{*e}}{3\omega_{de} - \tfrac{9}{5}(1+\eta)\omega_{*e}}, \tag{33}$$

which is marginally stable when $\omega_{de}/\omega_{*e} = (7-3\eta)/5$. Furthermore, if we relax the flute requirement and permit $\overline{\phi^2} > \bar{\phi}^2$ we find that a nonflutelike eigenfunction is more stable than a flutelike ($\overline{\phi^2} = \bar{\phi}^2$) mode. For $\kappa_i \neq 0$ the MHD and drift branches are coupled.

We will first investigate the dependence of the interchange mode on the plasma profile factor, η. In Fig. 1 we

FIG. 1. Normalized pressure gradient $\hat{\omega}_{*p}/\hat{\omega}_d$ vs the profile factor $\eta = n\nabla T/T\nabla n$ for $\kappa_i = 0.01$ (solid) and $\kappa_i = 0.1$ (dashed curve).

FIG. 2. The imaginary (solid) and real (dashed curve) frequency (ω/ω_{*e}) for the unstable modes obtained when $\hat{\omega}_{*p}/\hat{\omega}_d = 5/3$.

plot ω_{*p}/ω_d vs η for small $\kappa_i = 0.01$ and for finite $\kappa_i = 0.1$. The areas below the curves are stable. Consider first the stability boundary in the small Larmor radius limit, i.e., $(k_\perp \rho_i)^2 = 0.01$. MHD theory requires $\hat{\omega}_{*p}/\hat{\omega}_d < 5/3$ for stability and we observe the identical limit from kinetic theory when $\eta = 2/3$. As η deviates from 2/3, however, the stability requirements become more stringent. In Fig. 2 we plot the real and imaginary solution for the unstable modes obtained when $\hat{\omega}_{*p}/\hat{\omega}_d = 5/3$ for $K_i = 0.01$. We observe that when $\eta < 2/3$ the mode has a real frequency that is near to zero and a growth rate $\gamma/\hat{\omega}_{*e} \sim 5$ to 10. In the large η limit, i.e., for $\eta > 2/3$, we obtain a pair of drift waves with $\omega \sim \gamma > \hat{\omega}_d$ which are driven unstable by the temperature gradient. These two modes propagate in both the ion and the electron diamagnetic directions.

Figure 3 displays the real frequency at the $\kappa = 0.01$ marginal stability boundary of Fig. 1. Again we observe that the

FIG. 3. The real frequency (ω/ω_{*e}) at the marginal stability boundary for $\kappa_i = 0.01$.

real mode frequency is near zero for $\eta < 2/3$ and is determined by the drift frequency for $\eta > 2/3$. The significance of $\eta = 2/3$ is that for an adiabatic exchange of flux tubes the temperature and density profiles are left unchanged at this value of η. Thus there is no added instability drive due to temperature-gradient effects and this is the most stable operating point.

Figure 1 also shows the stabilizing effects of finite Larmor radius. We observe that for the FLR factor $\kappa_i = 0.1$ the stable region is increased for the MHD-like mode and we should expect the lowest toroidal mode number mode ($m = 1$) to be the most unstable mode, as it is least effected by FLR. When $\eta > 2/3$, FLR is not observed to be stabilizing.

MHD flute modes will coalesce with electrostatic interchange modes in the low beta limit because, for flute modes, neither the electrostatic potential nor $\mathbf{B}_\perp$ vary along a field line and therefore $\mathbf{E}_\parallel = 0$ and the modes satisfy the MHD assumption.[9] However, we have shown that the character of interchange modes that are stabilized by compressibility in closed field line systems is quite different from what MHD theory would predict. The stability boundary agrees with the MHD prediction when $\eta = 2/3$, but becomes more restrictive when $\eta \neq 2/3$. Near marginal stability the MHD mode is coupled to a temperature-gradient-driven mode and the real and imaginary frequencies are of the order of the drift frequency.

At high β, flute modes will have $\tilde{B}_\parallel \neq 0$ and MHD usually indicates that unstable modes have a ballooning character at high beta. It has been shown, however, that in a dipole field with the pressure profile chosen to be marginally stable to interchange modes, the lowest order odd ballooning mode and all higher modes are stable when the interchange mode is marginally stable.[10,11] Furthermore it has been shown that in this situation the lowest order even mode is stable when the interchange mode is stable and vice versa[3] and in fact at marginal stability this mode is the interchange mode. The stability of high beta interchange modes will be treated in a future publication.

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